An intuitive interpretation of the Fourier Transform

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Let's start with a reminder on the dot product, which we will refer to as scalar product. If we take a vector $\vec{v}$ in $\mathbb{R}^n$, and a reference vector $\vec{e}$, the scalar product of $\vec{v}$ and $\vec{e}$, defined as

$$\langle \vec{v}, \vec{e} \rangle = \sum v_i e_i, \tag{1}$$

describes the projection of $\vec{v}$ onto $\vec{e}$. If we take a set of reference vectors $\{\vec{e}^{(1)} \ldots, \vec{e}^{(n)}\}$ forming a base of $\mathbb{R}^n$, then in that base we have

$$\vec{v} = \left(\langle \vec{v}, \vec{e}^{(1)} \rangle, \ldots, \langle \vec{v}, \vec{e}^{(n)} \rangle \right). \tag{2}$$

In other words, aside from its "natural" representation $\vec{v} = (v_1, v_2, \ldots, v_n)$, $\vec{v}$ can also be represented in any base by its projection onto the vectors forming that base.

Let's now look at functions. One can see a function as a vector, although the space it lives in is much larger than where regular vectors live[1]. Now one important result: given two functions f and g, the operation

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x)\overline{g(x)}dx \tag{3}$$

is a scalar product.[2] ($\overline{g(x)}$ means complex conjugate.) Therefore, aside from the usual representation of function f through the number it associates to each possible argument x, one can equivalently "describe" f through its projection onto a set of other functions forming a base of the space of functions:

$$f = \{\langle f, g_\alpha \rangle \ldots \tag{4}$$

Importantly, a base of the space of functions would not have a finite number of functions, but an infinite (uncountable) one... so it could be represented, for instance, by a function!

[1]Essentially, because every number of the real line (or complex plane) is a possible "direction" of the space of functions.

[2]This indeed appears as the natural extension of the discrete scalar product of regular vectors in Eq. 1, if we consider each x as a component vector component.

With that being said, we only need to know now that: the set of functions $h_\omega(x) = e^{i\omega x}$, for all $\omega \in \mathbb{R}$, forms a base of the space of functions. Therefore, f can be represented by its projection onto each of these functions:

$$f = \{\langle f, e^{i\omega_1 x}\rangle \dots \tag{5}$$

And because the ω cover the entire line of reals, and for each ω there is a corresponding value of the scalar product, the set of all projections actually defines a *function* of ω. So in the base of the $e^{i\omega x}$, the functions f can be represented by the set of its projections, which is a new function $\hat{f}(\omega)$ which for every $\omega \in \mathbb{R}$ associates the projection:

$$\int_{-\infty}^{+\infty} f(x)e^{-i\omega x}dx.^3 \tag{6}$$

The function $\hat{f}(\omega)$ is called Fourier Transform of f, and provides a different representation of the same object.

The choice of this specific base of the space of functions, $e^{i\omega x}$, is convenient because it provides a representation of f in terms of sine and cosine waves, which is useful in a lot of situation. This is not, however, the only possible choice for a base of the space of functions. Alternative bases can be used, which can emphasize different features buried in f. For example, the theory of wavelet decomposition uses base functions that localized in space to detect more localized features of f.

[3]Note the minus sign: $e^{-i\omega x}$ is the complex conjugate of $e^{i\omega x}$